A novel energy efficient broadcast leader election

Philippe Jacquet and Dimitris Milioris
Alcatel-Lucent Bell Labs
Centre de Villarceaux
91620, Nozay, France
Email: {philippe.jacquet, dimitrios.milioris}@alcatel-lucent.com

Paul Mühlethaler
INRIA
Paris-Rocquencourt
BP 105. 78153 Le Chesnay Cedex
Email: paul.muhlethaler@inria.fr

***Abstract*—We introduce a new algorithm to achieve a distributed leader election in a broadcast channel that is more efficient than the classic Part-and-Try algorithm. The algorithm has the adavantage of having a reduced overhead $\log\log N$ rather than $\log N$. More importantly the algorithm has the a greatly reduced energy consumption since it requires $O(N^{1/k})$ burst transmissions instead of $O(N/k)$, per election, k being a parameter depending on the physical properties of the medium of communication. The algorithm has interesting potential applications in wireless cognitive networking.**

I. Introduction

Leader Election is the name given to a class of distributed algorithms that enable the random selection of one winner (the leader) among $n \leq N$ contenders over a maximum population of N users [1] [6], [10]. Leader election algorithms have applications in telecommunication, distributed databases, *etc*, the key point being the characterization of the medium of communication. There are many possibilities in the characterization of the communication medium, the pioneering cases were specified over a *ring* network. Here we assume that the communication medium is of the *broadcast* type and is prone to collisions, this case being more suitable for wireless applications [2]. To simplify our presentation we assume that the time is slotted. A slot can be either

- empty, the slot does not contain any burst.
- collision, the slot contains at least two burst that are in collision and they are not decodable.
- successful, the slot contains a single burst without collision.

The principle of leader election in a collision network has numerous origins. To the best of our knowledge the first description of such leader election was in [2] where the n contenders transmit bursts in slots with probability $\frac{1}{n}$ and the first successful transmission becomes the leader. This makes an $O(1)$ time for election together with a $O(1)$ global energy cost (to be defined later). Unfortunately to set up the probability of burst transmission, contenders need to be aware of their number, which requires a collision multiplicity estimator algorithm. If the estimate is too large, *e.g.* set at N the connected population size, then for small n the time for the election will be in $O(N)$ with a global energy cost still in $O(1)$. On the other hand, if the estimate is too small, *e.g.* set at 2, then the average time for the election increases to $\frac{1}{n}2^n$ with an average energy cost of $O(2^n)$. The *Part-and-Try* algorithm [3] does the multiplicity estimation and the election as well. It works as follows. Every contender has a fair coin. At the first slot every winner (*i.e.* those who tossed heads) transmits a burst that contains its identification. The losers (those who tossed tails) are listening during the slot and are eliminated if they hear the burst. For the second slot the contenders who have not been eliminated toss their coins again and repeat the process, which is repeated until a slot contain a successful burst. The transmitter of this last burst is the leader.

Many studies have been carried out on the Part-and-Try election process [4]. For example, the duration of the reduction phase, *i.e.* the phase between the first slot and the first collisionless slot (either empty or successful) is shown to be in average $L_n = \log_2 n + O(1)$. The number of surviving contenders of the reduction phase is $r_n = O(1)$ (in fact close to $\frac{1}{\log 2}$ and the remaining phase that achieves the leader election is also $O(1)$.

By *global* energy cost we mean the total cumulated cost of burst transmissions to get an election. This definition differs from the energy cost defined in [11] where only the energy cost of the winner, not the global energy cost which is more appropriate for wireless networks. To the author best knowledge, the leader election or the collision resolution algorithms have never been investigated under the total energy cost aspect. Considering the global energy cost, one thus must weight each burst with the number of actual transmitters in the slot. If 100 contenders transmit one burst in one slot, then the global energy cost of the slot is 100 burst. In this case the actual number of cumulated burst transmissions per election is on average equal to $E_n = n + O(1)$. It is clear that when n is of the order of several millions this becomes overwhelming. In particular, when considering a leader election in a wireless network, a burst transmission with such an energy would infer an interference range well beyond the area occupied by the network.

The main variant of the Part-and-Try election consists of introducing biased coin tossing. If $q = 1 - p$ is probability of head, then the average duration of a reduction phase becomes $L_n = \frac{\log n}{\log(\frac{1}{q})} + O(1)$ and the average number of surviving contenders becomes close to $\frac{p}{q\log(\frac{1}{q})}$. The latter diminishes when the former increases. Meanwhile the average global energy skyrockets to $\frac{q}{p}n + O(1)$ burst transmissions when $p \to 0$.

In this paper, we introduce a scheme, called *leader green election* (LGE) that reduces the reduction phase to $L_n = O(k \log_k \log N)$ when N is the size of the network and $k \geq 2$ is a parameter of the algorithm. More importantly the scheme reduces the actual global energy cost per election into $E_n = O(N^{1/k} \log_k \log N)$. The scheme is so efficient that it can be used in a repetitive way as the basis for the medium access scheme in cognitive WiFi networks. Figure 9 illustrates the average energy cost for the new algorithm versus the classic part and try algorithm. In fact we limit the description of our algorithm to the reduction phase, that we call election the phase. With this new wording an election phase can be either *successful* if the final burst is a success or *failed* if the final burst is a collision. When a failure occurs, the recommended procedure is to restart a new election phase and repeat until it is successful. The low and bounded residual collision rate would make this happen in $O(1)$ time.

The LGE scheme seems inappropriate to be applied sensor networks. It needs a slot synchronization and full connectivity (at least toward an access point, thus not suitable for cluster head election). Most of the literature about energy saving in sensor networks [14], [15], [16], [17], [17], [18], [19] have concern with cluster organization in a multi-hop topology or with the introduction of technical twists in the physical layer.

The paper is organized as follows. In the next section we describe the LGE algorithm. Then in the following section we analyze the performance of the LGE algorithm. We split the performance section into two parts: one part devoted to the residual survivor collision rate, and the other part to the energy costs. In a separate section, we describe the application of LGA algorithm to a primary protocol in cognitive WiFi. We also devote a section to numerical simulations.

II. The LGE algorithm

As a generalization of the broadcast leader election, we assume that at the beginning of the election phase every contenders compute a binary election key. During the election phase (reduced to the reduction phase), every contender will schedule a burst transmission according to its election key. In short, if there is a zero on the ith bit of its election key, and assuming that the contender has survived the election phase until this ith slot, then the contender will transmit no burst during this slot, otherwise the contender (if surviving) will transmit a burst. For example the part and try leader election leads to the fact the election key of a contender is a (potentially) infinite sequence of i.id. bits with uniform distribution on $\{0, 1\}$.

For our new algorithm we assume that the broadcast medium has N connected users (assume $N \approx 10^6$) and that the number of contenders n is always smaller or equal to N. We assume that the integer k is fixed (*e.g.* $k = 10$) and that there exist a number L_N function of N. We will have $L_N = O(\log_k \log N)$ (for $N = 10^6$, $k = 10$ and we would have $L_N = 3$). We also fix a number p between 0 and 1, used in all terminals, we assume that p is not close to one (*e.g.* $p = 0.02$).

For the rest of the paper we define the set of k binary super symbols $\mathcal{A}_k = \{B_0, \dots, B_{k-1}\}$. The super-symbol B_ℓ, with $\ell < k$, is $k - \ell - 1$ 0's followed by a 1, or in short $B_\ell = 0^{k-\ell-1}1$.

The election key of each contender is made up of L_N super-symbols as follows. Every contending terminal independently select an integer X with a geometric distribution of probability rate p uniform for all terminals. We have $P(X = m) = pq^m$ and $P(X \geq m) = q^m$ with $q = 1 - p$. The key encoding is the following:

- If $X \geq k^{L_N}$, then $S = B_{k-1} \cdots B_{k-1}$;
- otherwise S is the encoding in base k of X where B_0 corresponds to 0, B_1 corresponds to 1, B_ℓ corresponds to ℓ.

For example when $X = 0$ then the election key is $B_0 \cdots B_0$, 1 in $B_0 \cdots B_0 B_1$ and $k^{L_N} - 1$ in $B_{k-1} \cdots B_{k-1}$. In short the key is equal to the encoding of $\min\{X, k^{L_N} - 1\}$ in base k with the super alphabet.

Equivalently, that if $S_{L_N} S_{L_N - 1} \cdots S_1$ are the L_N super symbol of the election key, then the S_j are independent for $j \in \{1, \dots, L_N\}$ and

$$P(S_j = B_\ell) = q^{\ell k^{j-1}} \frac{1 - q^{k^{j-1}}}{1 - q^{k^j}} \tag{1}$$

III. Performance analysis

A. LGE Collision rate

In this section we investigate the probability that more than two contenders will survive the green election phase. We have the obvious lemma:

Lemma 1. *the survivor collision rate is smaller than $r_n - 1$ where r_n is the average number of surviving contenders when the election phase starts with $n \leq N$ contenders*

We show in theorem 1 the bound

$$r_n \leq r_{1,n} + r_{2,n}$$

where $r_{1,n}$ is the average number of contenders that have selected $X \geq k^{L_N}$, and $r_{2,n}$ the average number of contenders that have selected $X = \bar{X}_n$ where $\bar{X}_n$ is the maximum value of the random variables X selected by the n contenders.

Lemma 2 (Overflow rate). *In the election phase with $n \leq N$ contenders we have*

$$r_{1,n} \leq n q^{k^{L_N}} \ .$$

Proof: Each terminal has probability $q^{k^{L_N}}$ of having its integer X greater than k^{L_N}. ∎

Since $q < e^{-p}$, we notice that we only need $L_N \geq \log_k \left(\frac{1}{p} \log \frac{N}{\varepsilon} \right)$ in order to have the overflow rate smaller than ε.

Let J_n be the (random) set of contenders that have selected $X\bar{X}_n$. We have $r_{2,n} = E(|J_n|)$.

Lemma 3. *We have the asymptotic evaluation:*

$$r_{2,n} \leq \frac{-p}{q \log q} \sum_{m \in \mathbb{Z}} \Gamma\left(1 + \frac{2im\pi}{\log q}\right) n^{2im\pi/\log q} + O(n^{-1}) \ .$$

Remark:: the terms in $\Gamma\left(1 + \frac{2im\pi}{\log q}\right) n^{2im\pi/\log q}$ introduce a periodic contribution in $\log n$ with period $\log \frac{1}{q}$. Using the classic evaluation $|\Gamma(x+iy)| = O(\exp(-\pi|y|/2))$, when $|y| \to \infty$, we get the evaluation:

$$r_{2,n} \leq \frac{-p}{q \log q} + O\left(\exp(\frac{\pi^2}{\log q})\right) \ . \tag{2}$$

For $p < 0.1$ the periodic contribution is less than 10^{-40} and therefore the approximation $r_{2,n} = \frac{-p}{q \log q} = 1 + p/2 + O(p^2)$ suffices when n is large.

Proof: We have $r_{2,0} = 0$ and $r_{2,1} = 1$. Using classic combinatoric analysis [12] we have the recursion:

$$r_{2,n} = np^n + \sum_{m>0} \binom{n}{m} q^m p^{n-m} r_{2,m} \tag{3}$$

which basically states that either all the n surviving contenders have produced $X = 0$ (with probability p^n) or the process continues on $X - 1$ with the m surviving contenders (with binomial probability $\binom{n}{m} q^m p^{n-m}$). Introducing the Poisson generating function

$$R_2(z) = \sum_n r_{2,n} \frac{z^n}{n!} e^{-z} \ , \tag{4}$$

we get the functional equation

$$R_2(z) = pze^{-qz} + R(qz) \ , \tag{5}$$

and the explicit solution as an harmonic sum:

$$R_2(z) = \sum_{m=0}^{m=\infty} pq^m ze^{-q^{m+1}z} \ . \tag{6}$$

Therefore

$$r_{2,n} = \sum_{m=0}^{m=\infty} pq^m n(1 - q^{m+1})^{n-1} \ . \tag{7}$$

Taking the bound $(1 - x) \leq e^{-x}$ we get

$$r_{2,n} \leq R_2(n-1)(1 + \frac{1}{n-1}) \ . \tag{8}$$

In fact a more thorough analysis using depoissonization [8] would lead to the identity $r_{2,n} = R_2(n)(1 + O(n^{-1}))$.

To get the asymptotics of $R_2(z)$ when $z \to \infty$ we use the Mellin transform of the function ze^{-z} which is $\int_0^\infty z^s e^{-z} dz = \Gamma(s+1)$, where the classic Euler *Gamma* function is analytic for $\Re(s) > -1$. By virtue of the analysis of harmonic sums, the Mellin transform of function $R_2(z)$ is:

$$\begin{aligned} R_2^*(s) &= \int_0^\infty z^{s-1} R_2(z) dz \\ &= \sum_{m=0}^{m=\infty} \int_0^\infty pq^m z^s e^{-q^{m+1}z} dz \\ &= \sum_{m=0}^{m=\infty} pq^{-s-1} q^{-ms} \Gamma(s+1) \\ &= \frac{pq^{-s-1}}{1 - q^{-s}} \Gamma(1+s) \ , \end{aligned} \tag{9}$$

which is analytic for any complex number s such that $-1 < \Re(s) < 0$. By reversing the Mellin transform we have

$$R_2(z) = \frac{1}{2i\pi} \int_{\Re(s)=c} z^{-s} R_2^*(s) ds$$

for all $-1 < c < 0$. By moving the integration line toward positive half plan we meet the simple poles of $R^*(s)$ which are at $s = \frac{2im\pi}{\log q}$ for $m \in \mathbb{Z}$ with respective residues $\frac{p}{q \log q} \Gamma\left(1 + \frac{2im\pi}{\log q}\right) n^{2im\pi/\log q}$:

$$\begin{aligned} R_2(z) &= \sum_{m \in \mathbb{Z}} \Gamma\left(1 + \frac{2im\pi}{\log q}\right) z^{2im\pi/\log q} \\ &\quad + \frac{1}{2i\pi} \int_{\Re(s)=M} z^{-s} R^*(s) ds \end{aligned} \tag{10}$$

for any $M > 0$. ■

Remark:: We can show via a similar method an estimate of the distribution of $|J_n|$. We show that for any complex number t:

$$\begin{aligned} E[e^{t|J_n|}] &= \frac{\log(1 - pe^t)}{\log(1-p)} + O(\frac{1}{n}) \\ &\quad + \text{negligible periodic terms} \ . \end{aligned} \tag{11}$$

Therefore all the moments are finite and bounded.

Theorem 1 (Collision rate)**.** *The collision rate of the contention scheme is smaller than*

$$Nq^{k^{L_N}} + \frac{-p}{q \log q} - 1 + O\left(\exp(\frac{\pi^2}{\log q})\right) \ . \tag{12}$$

Proof: Assume there are n contenders. Let $X_1, \ldots, X_n$ be respective values of their integer X. The winners of the contentions are those that either

- had their $X > k^{L_N}$;
- or $X = \bar{X}_n$.

Therefore their number is smaller than $N(1-p)^{k^{L_N}} + \frac{-p}{q \log q}$. Removing 1 (there is always a winner) gives an upper bound of the collision rate. ■

Clearly the collision rate can be made arbitrarily small by decreasing p and increasing L_N (but keeping it in $\log_k \frac{1}{p} \log \frac{N}{\varepsilon}$). Figure 6 shows the actual value of r_n versus the upper bound for $p = 0.02$, $k = 10$ and $L_N = 3$, for n from 1 to 10^6.

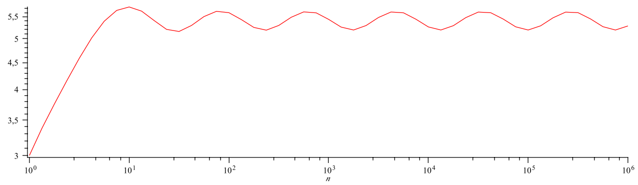

Fig. 1. Average global energy cost versus n from 1 to 10^6, with $k = 10$, $q = 0.98$ and $L_N = 3$.

B. LGE Energy Cost

We denote by $C(n)$ the global energy cost of an election with n contenders. Since the election may fail due to collision, the average energy cost per *successful* election E_n is larger than $E(C(n))$. To get an upper bound we have the following lemma:

Lemma 4. *We have the inequality:*

$$E_n \leq \frac{E(C(n))}{2 - r_n} \ . \tag{13}$$

Proof: If we consider an infinite sequence of elections with n contenders, it forms a sequence of i.i.d. elections. The quantity E_n is identical to the average cumulated energy cost between two successive succesful election. Let s_n be the probability that an election is successful, thus by virtue of the election renewal process we have the identity:

$$E_n = \frac{E(C(n))}{s_n} \ . \tag{14}$$

Since $s_n \geq 1 - (r_n - 1)$ the lemma is proven. ■

For the following we assume that $q^{k^{L_N}} = \Theta(N^{-1})$, this is obtained with $L_N = \log_k \frac{1}{p} \log \frac{N}{\varepsilon}$. In the following we denote $\bar{N} = q^{-k^{L_N}}$ which is greater than N but still $O(N)$. In Figure 1 we display the global average energy cost of the LGE algorithm for $p = 0.02$, $k = 10$ and $L_N = 3$. The next sections are devoted to the methodology to analytically derive these evaluations.

1) Energy cost of the first super-symbol: Let ℓ be an integer between 0 and $k-1$. We suppose that there are $n \geq 1$ contenders. We consider the first symbol of the election key. We denote by $C_\ell(1, n)$ the number of contenders which have B_ℓ as the first symbol and which *actually* transmit their first burst. We actually have $C_\ell(1, n) = 0$ when

- there are no contenders whose first symbol is B_ℓ, or
- there are contenders whose first symbol is $B_{\ell'}$ for some $\ell' > \ell$.

Theorem 2. *The average value of $C_\ell(1, n))$ satisfies*

$$E(C_\ell(1, n)) = n p_\ell q_\ell^{n-1} \tag{15}$$

with

$$\left\{ \begin{array}{rl} p_{k-1} = \frac{1}{\bar{N}^{(k-1)/k}} & q_{k-1} = 1 \ , \ \textit{when } \ell < k-1\textit{:} \\ p_\ell = \frac{1}{\bar{N}^{\ell/k}} - \frac{1}{\bar{N}^{(\ell+1)/k}} & q_\ell = 1 - \frac{1}{\bar{N}^{(\ell+1)/k}} \ . \end{array} \right. \tag{16}$$

Proof: We suppose that there are n contenders, $n \leq N$. We concentrate on the first pulse, since for the other pulse the number of actual transmitters can only decrease. We first concentrate on the highest super-symbol B_{k-1}. The condition that a node has its first super-symbol equal to B_{k-1} is that

$$X \geq (1 - \frac{1}{k})k^{L_N} \ . \tag{17}$$

The probability of such an event p_{k-1} satisfies

$$p_{k-1} = q^{(1-\frac{1}{k})k^{L_N}} = \frac{1}{\bar{N}^{1-1/k}} \ . \tag{18}$$

Therefore the average number of actual transmitters of symbol B_{k-1} satisfies t

$$E(C_{k-1}(1, n)) = n p_{k-1} \leq N p_{k-1} \leq \frac{N}{\bar{N}} \bar{N}^{1/k} \ . \tag{19}$$

We now turn our attention to super-symbol B_{k-2}. The condition that the super-symbol B_{k-2} is actually emitted is that

$$(1 - \frac{2}{k})k^{L_N} \leq \bar{X}_n < (1 - \frac{1}{k})k^{L_N} \ . \tag{20}$$

The probability that $X < (1 - \frac{1}{k})k^{L_N}$ is q_{k-2}^n with

$$q_{k-2} = 1 - q^{(1-\frac{1}{k})k^{L_N}} = 1 - \frac{1}{\bar{N}^{1-1/k}} \ . \tag{21}$$

Given $\bar{X}_n < (1 - \frac{1}{k})k^{L_N}$ the conditional probability that a terminal transmits B_{k-2}, *i.e.* that $(1 - \frac{2}{k}) \leq X < (1 - \frac{1}{k})k^{L_N}$ is equal to $\frac{p_{k-2}}{q_{k-2}}$ with

$$p_{k-2} = q^{(1-\frac{2}{k})k^{L_N}} - q^{(1-\frac{1}{k})k^{L_N}} = \frac{1}{\bar{N}^{1-2/k}} - \frac{1}{\bar{N}^{1-1/k}} \ .$$

Therefore the average number of terminals which actually transmit the super-symbol B_{k-2} is equal to $n p_{k-2} q_{k-2}^{n-1}$. Concerning further super-symbol B_ℓ, with $\ell < k-1$ the average number of transmitters is $n p_\ell q_\ell^{n-1}$ with

$$p_\ell = \frac{1}{\bar{N}^{1-\frac{k-\ell}{k}}} - \frac{1}{\bar{N}^{1-\frac{k-\ell-1}{k}}} \tag{22}$$

$$q_\ell = 1 - \frac{1}{\bar{N}^{1-\frac{k-\ell-1}{k}}} \ . \tag{23}$$

and like for $\ell = k-2$, the average number of symbol B_ℓ transmitters is equal to $n p_\ell q_\ell^{n-1}$. ■

Theorem 3. *Let $\ell < k$, the average number of contenders transmitting B_ℓ as the first burst satisfies*

$$\left\{ \begin{array}{rcl} E(C_{k-1}(1, n)) & \leq & \frac{N}{\bar{N}} \bar{N}^{1/k} \ , \ \textit{when } \ell < k-1\textit{:} \\ E(C_\ell(1, n)) & \leq & \frac{1}{e} \bar{N}^{1/k} \ . \end{array} \right. \tag{24}$$

Proof: Indeed $E(C_\ell(1, n)) = n p_\ell q_\ell^{n-1}$. The special case $\ell = k-1$ is immediate. Let for $g(x) = x e^{-x} x$ for a real number x . For $\ell < k-1$ we have

$$\begin{aligned} n p_\ell q_\ell^{n-1} & \leq (\bar{N}^{1/k} - 1) \frac{n}{\bar{N}^{(\ell+1)/k}} \exp(-\frac{n-1}{\bar{N}^{(\ell+1)/k}}) \\ & \leq \bar{N}^{1/k} (1 - \frac{1}{\bar{N}^{1/k}}) \exp(\frac{1}{\bar{N}^{(\ell+1)/k}}) g\left(\frac{n}{\bar{N}^{(\ell+1)/k}}\right) \\ & \leq \bar{N}^{1/k} \exp((\frac{1}{\bar{N}^{(\ell+1)/k}} - \frac{1}{\bar{N}^{1/k}}) g\left(\frac{n}{\bar{N}^{(\ell+1)/k}}\right) \\ & \leq \bar{N}^{1/k} g\left(\frac{n}{\bar{N}^{(\ell+1)/k}}\right) \ . \end{aligned}$$

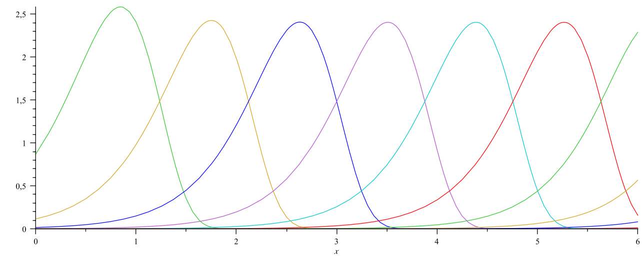

Fig. 2. $E(C_\ell(1,n))$ as a function of n (semilog plot). From left to right, different colors, for super-symbols B_0, B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8 and B_9 when $k=10$, $q=0.98$ and $L_N=3$.

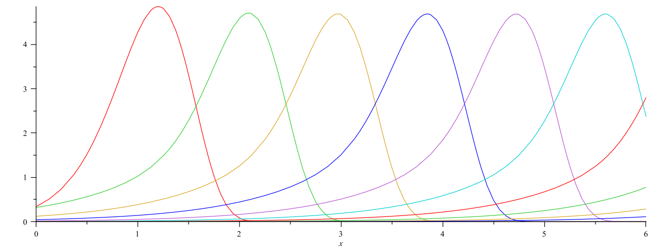

Fig. 3. Theoretical standard deviation $\sqrt{\text{var}(C_\ell(1,n))}$ versus n from 1 to 10^6 (semilog plot).From left to right, different colors, for super-symbols B_0, B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8 and B_9 when $k=10$, $q=0.98$ and $L_N=3$.

Since $\max_{x\geq 0}\{g(x)\}=\frac{1}{e}$ the theorem is proven. ■

Figure 2 shows the quantities $E(C_\ell(1,n)$ for $\ell=0,\ldots,9$ with n ranging from 1 to 10^6. We assume $q=0.98$, $k=10$ and $L_N=3$, thus $\bar{N}=5.941885894\times 10^8$. When n is small, say $n\leq 50$, the first super-symbol transmitted is most likely to be B_0 corresponding to the weakest preambles, beyond $n>50$ it would be B_1, then B_2, *etc*. We display the average number of transmitters with respect to the 10 different super-symbols B_ℓ with plots of different colors, ranked from left to right with decreasing ℓ. Notice that the numbers for B_2 and B_1 are smaller and almost unnoticeable for B_0, since $\bar{N}\gg N$.

Theorem 4. *Let t be a complex number, we have the identity:*

$$E(e^{tC_\ell(1,n)})=1-q_\ell^n+(q_\ell-p_\ell+p_\ell e^t)^n\ . \tag{25}$$

Consequently the moments are computable and in particular

$$var(C_\ell(1,n))=O(\bar{N}^{2/k})\ . \tag{26}$$

Proof: The expression comes from the fact that $C_\ell(n)$ is a compound Bernoulli variable:

- with probability $1-q_\ell^n$ it is zero;
- with probability q_ℓ^n it is a Bernoulli trial over n elements with individual probability $\frac{p_\ell}{q_\ell}$.

The second moment satisfies:

$$\begin{aligned}
E((C_\ell(1,n))^2) &= n(n-1)p_\ell^2 q_\ell^{n-2}+np_\ell q_\ell^{n-1}\\
&\leq n^2p_\ell^2q_\ell^{n-2}\\
&\leq \left(\frac{n}{\bar{N}^{\frac{\ell+1}{k}}}\right)^2(\bar{N}^{\frac{1}{k}}-1)^2\exp(-\frac{n-2}{\bar{N}^{\frac{\ell+1}{k}}})\\
&\leq \bar{N}^{\frac{2}{k}}g_2\left(\frac{n}{\bar{N}^{\frac{\ell+1}{k}}}\right)
\end{aligned}$$

with $g_2(x)=x^2e^{-x}$. Since $\max_{x\geq 0}\{g_2(x)\}=2e^{-\sqrt{2}}$, we have

$$E((C_\ell(1,n))^2)\leq 2e^{-\sqrt{2}}\bar{N}^{\frac{2}{k}}\ . \tag{27}$$

The variance has the expression:

$$\text{var}(C_\ell(1,n))=np_\ell q_\ell^{n-1}-np_\ell^2q_\ell^{n-2}+n^2p_\ell^2(q_\ell^{n-2}-q_\ell^{2n-2})\ , \tag{28}$$

and we have $\text{var}(C_\ell(1,n))=O(N^{2/k})$.

In Figure 3 we display the exact theoretical standard deviation of the number of transmitters on the first pulse for n varying from 1 to 10^6. Notice that the standard deviation is larger than the mean by a factor of around 2. ■

2) Energy cost of the first burst: We now look at the number of actual transmitter $C(1,n)$ of their first burst regardless of the symbol. Since the actual transmission of the first pulse of the preamble sequence can only be on the lexicographical largest symbol B_ℓ over all the preamble sequence in competition, $C(1,n)=\sum_\ell C_\ell(1,n)$.

Theorem 5. *Starting an election over n contenders, the average number of the contenders that actually transmit their first burst satisfies the inequality*

$$E(C(1,n))\leq\frac{n}{n-1}(\bar{N}^{1/k}-1)\frac{k}{\log\bar{N}}(A(k,\bar{N})+\bar{N}^{1/k}r_{1,n} \tag{29}$$

with

$$A(k,\bar{N})=\sum_{m\in\mathbb{Z}}\left|\Gamma(1+\frac{2ikm\pi}{\log\bar{N}})\right| \tag{30}$$

Proof: We start with some straighforward analytical bounds. For $\ell<k-1$ we already know that we have

$$np_\ell q_\ell^{n-1}\leq(\bar{N}^{1/k}-1)\frac{n}{n-1}g\left(\frac{n-1}{\bar{N}^{(\ell+1)/k}}\right)\ . \tag{31}$$

The case $\ell=k-1$ is specific since it gives $np_{k-1}=\frac{n}{\bar{N}^{(k-1)/k}}=\bar{N}^{1/k}r_{1,n}$. Therefore

$$\begin{aligned}
E(C(1,n))-\bar{N}^{1/k}r_{1,n} &\leq (\bar{N}^{1/k}-1)\frac{n}{n-1}\sum_{\ell=1}^{k-1}g(\frac{n-1}{\bar{N}^{\ell/k}})\\
&\leq (\bar{N}^{1/k}-1)\frac{n}{n-1}\sum_{\ell\in\mathbb{Z}}g(\frac{n-1}{\bar{N}^{\ell/k}})\ .
\end{aligned}$$

Let $G(x)=\sum_{\ell\in\mathbb{Z}}g(\frac{x}{\bar{N}^{\ell/k}})$. It turns out that $G(e^x)$ is periodic of period $\frac{1}{k}\log\bar{N}$. Using the Fourier transform:

$$G(e^x)=\sum_{m\in\mathbb{Z}}\gamma_m e^{2imk\pi x/\log\bar{N}}\ , \tag{32}$$

with

$$\begin{aligned}
\gamma_m &= \frac{k}{\log\bar{N}}\int_0^{\frac{1}{k}\log\bar{N}}G(e^x)e^{2i\pi mkx/\log\bar{N}}dx\\
&= \frac{k}{\log\bar{N}}\sum_{\ell\in\mathbb{Z}}\int_{\bar{N}^{\ell/k}}^{\bar{N}^{(\ell+1)/k}}g(y)y^{2i\pi mkx/\log\bar{N}}dy\\
&= \frac{k}{\log\bar{N}}\int_0^\infty g(y)y^{2i\pi mkx/\log\bar{N}}dy=\Gamma\left(1+\frac{2i\pi mk}{\log\bar{N}}\right)\ .
\end{aligned}$$

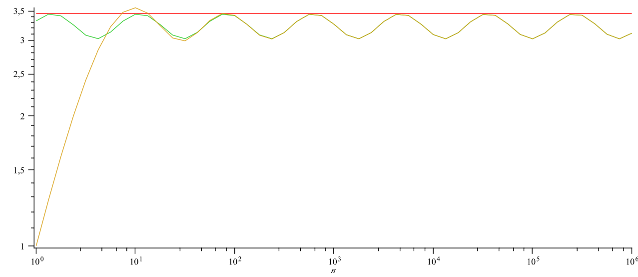

Fig. 4. First burst emitters. Theoretical $E(C(1,n))$ versus the number of initial contenders n (semilog plot). Exact theoretical (brown), function $(\bar{N}^{1/k}-1)\frac{k}{\log \bar{N}}G(n)$ (green), quantity $(\bar{N}^{1/k}-1)\frac{k}{\log \bar{N}}A(k,\bar{N})$ for $k=10$ and $L_N=3$.

we get

$$G(x)=\sum_{m\in\mathbb{Z}}\frac{k}{\log \bar{N}}x^{2ikm\pi/\log \bar{N}}\Gamma\left(1+\frac{2ikm\pi}{\log \bar{N}}\right) \qquad (33)$$

and the bound $|G(x)|\leq \frac{k}{\log \bar{N}}A(k,\bar{N})$ naturally appears. This terminates the proof. Figure 4 displays the various bounds versus actual $E(C(1,n))$. The display does not match with the upper bound for low value of n since the factor $\frac{n}{n-1}$ is omitted. ■

Theorem 6. *Let t be a complex number. The quantity $E(e^{tC(1,n)})$ satisfies the identity:*

$$E(e^{tC(1,n)})=\sum_{\ell=0}^{\ell=k-1}(q_\ell-p_\ell+p_\ell e^t)^n-(q_\ell-p_\ell)^n\ . \qquad (34)$$

and all the moments are computable. In particular var$(C(1,n))=O(\bar{N}^{2/k})$.

Proof: We know that $C(1,n)=\sum_{\ell=0}^{k-1}C_\ell(1,n)$ but since always $C(1,n)\geq 1$ and there are never more than one $C_\ell(1,n)\neq 0$, we have

$$E(e^{tC(1,n)})=\sum_{\ell=0}^{\ell=k-1}E(e^{tC_\ell(1,n)}\min\{1,C_\ell(1,n)\}) \qquad (35)$$

since $\min\{1,C_\ell(1,n)\})=1$ if $C_\ell(1,n)\geq 1$ and 0 otherwise. We have

$$E(e^{tC_\ell(1,n)}\min\{1,C_\ell(1,n)\})=(q_\ell-p_\ell+p_\ell e^t)^n-(q_\ell-p_\ell)^n \qquad (36)$$

which terminates the proof of the identity. Incidentally we notice for $\ell\geq 1$ $q_\ell-p_\ell=q_{\ell-1}$. The result on thew variance comes from the fact that

$$E((C(1,n))^2)=\sum_{\ell=0}^{k-1}E((C_\ell(1,n))^2) \qquad (37)$$

and that $E((C_\ell(1,n))^2)=O(\bar{N}^{2/k})$. See the illustration in figure 14. ■

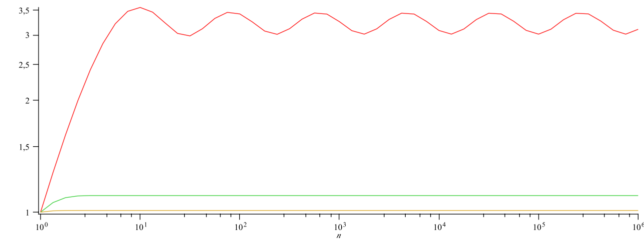

Fig. 5. Theoretical $E(C(1,n))$ (red), $E(C(2,n))$ (green) and $E(C(3,n))$ (brown) versus n from 1 to 10^6, with $k=10$, $q=0.98$ and $L_N=3$.

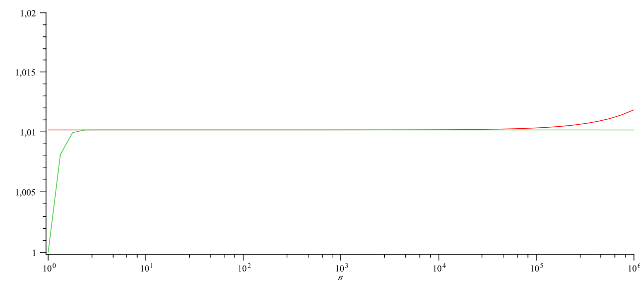

Fig. 6. Theoretical $r_n-1=E(C(L_N,n))-1$ (green) and upper bound $r_{1,n}+r_{2,n}-1$ versus n from 1 to 10^6, with $k=10$, $q=0.98$ and $L_N=3$.

3) Cumulated energy cost per election phase: Let j be an integer between 1 and L_N, we denote by $C(j,n)$ the actual number of survivors that actually transmit their jth burst, given that the election started with $n\leq N$ contenders. We know that $C(j,n)\leq C(i,n)$ in distribution when $j\geq i$. We have the additional estimate that basically illustrates that $E(C(j,n))=O(N^{1/k^j})$.

Theorem 7. *We have $E(C(n))=\sum_{j=1}^{L_N}E(C(j,n))$ with:*

$$E(C(j,n))\leq\frac{n}{n-1}(\bar{N}^{1/k^j}-1)\frac{k^j}{\log \bar{N}}A_j(k,\bar{N})+\bar{N}^{1/k^j}r_{1,n} \qquad (38)$$

with

$$A_j(k,\bar{N})=\sum_{m\in\mathbb{Z}}\left|\Gamma(1+\frac{2ik^jm\pi}{\log \bar{N}})\right|\ . \qquad (39)$$

Notice that $(\bar{N}^{1/k^j}-1)\frac{k^j}{\log \bar{N}}\to 1$ when $j\to\infty$.

Proof: The proof is the same as the previous proofs with the difference that we have to consider the super alphabet $(\mathcal{A}_k)^j$ instead of $\mathcal{A}_k$ to support the j first super-symbols of the election key: $S_{L_N}\cdots S_{L_N-j+1}$. Therefore it suffices to replace all instances of k by k^j in the analysis.

Figure 5 shows the different values of $E(C(j,n)$, notice that $E(C(L_N,n)=r_n$ the residual collision rate of the survivors. Figure 6 shows $r_n-1=C(L_N,n)$ with the theoretical upper bound $r_{1,n}+r_{2,n}-1$. ■

C. Performance generalization

The above methodology allows us to extend our analysis to cases where L_N is larger or smaller than 3, or when p is larger or smaller than 0.02. In Figure 7 we display the upper bound of the collision rate for different values of p and L_N for

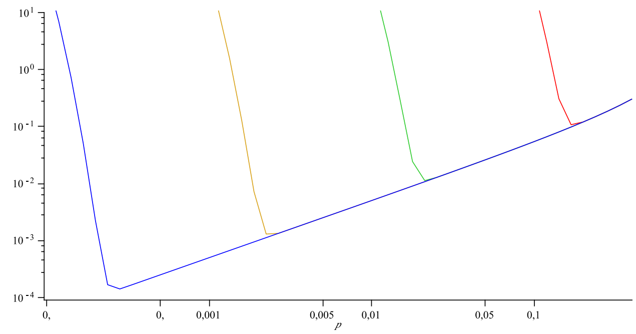

Fig. 7. Theoretical upper bounds of collision rates for $k = 10$ and $n = 10^6$ versus p. From right to left $L_N = 2, 3, 4, 5$.

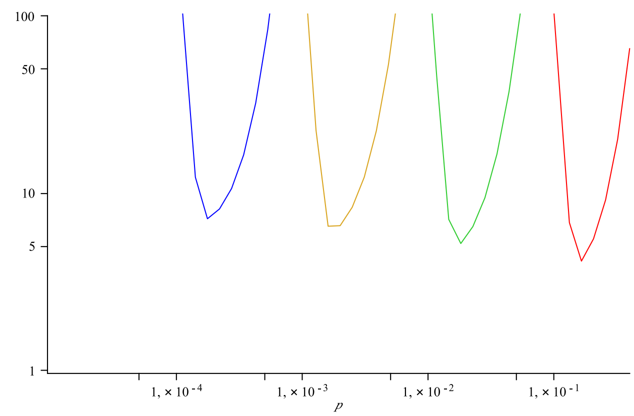

Fig. 8. Theoretical upper bounds of energy costs for $k = 10$ and $n = 10^6$ versus p. From right to left $L_N = 2, 3, 4, 5$.

$k = 10$ and t$n = 10^6$. Figure 8 displays the average energy cost in the same conditions. In both case we use the tight upper bounds obtained in previous sections, since the exact formulas would be difficult to handle (for $L_N = 5$ we would need to handle 10^5 terms).

The sharp shapes in both figures (most dramatic for the energy costs) come from the fact that when p decreases both r_n and $E(C(n))$ jumps since the probability that individual random variable X exceeds k^{L_N} tends to 1, and therefore $r_n \to n$ and $E(C(n)) \to kn$.

Indeed when n and L_N are large we have the equivalence

$$E(C(n)) \sim (q^{-k^{L_N-1}} - 1)\frac{1}{k^{L_N-1}\log\frac{1}{q}} + nq^{k^{L_N}} \quad . \tag{40}$$

Noting $y = k^{L_N-1}\log\frac{1}{q}$ the formula becomes $\frac{e^x-1}{x} + ne^{-kx}$. It is optimized on $x = \frac{\log kn}{k+1}(1 + O(\frac{1}{\log_k \log n}))$ thus $L_N = O(\log_k \log n)$ and

$$E(C(n)) = (nO(\log n))^{1/(k+1)} \quad . \tag{41}$$

For example for $n = 10^{12}$ the minimum is attained with $x \approx 2.853$ and $E(C(n)) \approx 6.134$. This can be achieved with $L = 3$ and $p \approx 2.853.10^{-2}$. In fact exact computation shows $E(C(n)) \approx 15.07$ since the convergence in $\frac{1}{\log_k \log n}$ is very slow.

IV. LGE in cognitive WiFi

One of the applications of green election is for wireless collision algorithms in particular in cognitive wireless networks where the secondary network is WiFi IEEE 802.11 [7]. Since the green election is low energy consuming, it can be used as a systematic and repetitive medium access control that will naturally prevail over the WiFi CSMA scheme.

The objective is to enable the primary user of a cognitive network to use bursty access to the medium so that any primary burst will be separated by a time interval smaller than the standard time spacing (DIFS) in WiFi. That way the primary user will pre-empt the use of the network by secondary WiFi user. The pre-emption will be active as long as there is primary traffic. In the absence of primary traffic, no burst is transmitted and the secondary WiFi traffic takes the medium.

In [13] we describe a scheme where primary devices transmit a preamble signals of bursts before transmitting their packet. This preamble or so-called "comb" is made up of on-off transmissions in mini time slot times. A terminal consider that a mini slot is "on" if it transmits a burst during it. The time-slot is "off" if it does not transmit a burst and instead listens to the channel. The contention algorithm is such that if a terminal detects a burst transmitted by another terminal during one of its own off periods, then it immediately aborts its preamble transmission and defers for the next contention phase. The detection of bursts come by simply tracking the energy level on the carrier. There is a mapping between the bursty preambles and binary sequences, by just reading an off period as a 0, and an on period as a 1. The winners of the contention are the terminals which have the largest preamble sequence in lexicographic order. In the rest of the paper we assume that there are N nodes in the network.

In order to make the bursty preamble pre-empt any secondary network operating under WiFi, the preamble binary sequences must be such that bursts are never separated by more than k mini-slots. The integer k is the ratio between the wifi DIFS slot interval and the mini-slot duration. We assume that k is of the order of 10. In [13] we introduce a scheme where nodes' preamble sequences are mapped from their identification numbers translated into the super-alphabet $\mathcal{A}_k = \{B_0, B_1, \ldots, B_k\}$. It turns out that the length of the preamble must be in $\log N$. The major issue with the scheme described in [13] is that when all the N nodes contend in the same resolution epoch, then the first symbol of the preamble, which is likely B_{k-1} must be simultaneously transmitted by around $\frac{N}{k}$ nodes[1]. If $N = 10^6$ the increase in power would be of 50 dB, consequently the first slot would create interference far beyond the individual radio range of the nodes, and therefore damage the communication well ouside this range. If we assume that the attenuation factor of wave propagation is 2, then the instantaneous interference radius will be around $1,000$

[1]In fact, it is larger, since the optimal ID translation is when the symbol B_ℓ is affected with probability $\rho^{\ell+1}$ where ρ satisfies $\sum_{\ell \leq k} \rho^{1+\ell} = 1$, therefore $\rho > \frac{1}{k}$

times the individual radio range, *i.e.* for a 100 m individual radio range the interference range could be 100 km which is not acceptable. Furthermore nodes will transmit their first symbols $50,000$ ($\frac{N}{2k}$) times on average per packet. The second pulse would be transmitted in average $5,000$ times, the m pulse $\frac{N}{2k^m}$ times. This would incur a cost in energy of several orders of magnitude greater than the transmission cost of the packet itself.

In fact making all preamble sequences different is quite an unnecessary requirement. In theory, there is no need to guarantee in theory a collision-free contention scheme, since there is always an incompressible loss rate due to the random nature of radio. Therefore we can use the broadcast green leader election and achive an arbitrary close to 0 collision rate (in theory, omitting radio noise). If a collision occurs, then the terminals involved will simply retransmit in the next contending phase. The low collision rate guarantees a successful transmission in, on average, a very small number of retries. Furthermore the scheme has the very interesting property that the average number of simultaneous transmitters of a signaling burst is smaller than $O(N^{1/k})$. For $k = 10$ and $N = 10^6$ this would give an l average of 4 at most, making a signal increase of 6 dB per burst, well in the limit of a classic random fading. With attenuation factor 2, the interference range would only double occasionally. On the energy saving plan, a node would not need to transmit more, in average, than 4 (in length, $N^{1/k}$) times each pulse before transmitting its own packet.

For the numerical experimentation (next section) we have considered that the bound $N = 10^6$ is well absorbed by a green election with $L_N = 3$ super-symbols, and probability $p = 0.02$. In this case the residual collision rate is around 10^{-2}, with an average burst transmission cost around 5.6. However the energy cost should also include the cost of the packet transmitted in a collision, *i.e.* a cost of 10^{-2} extra cost on a packet transmission cost unit.

We also study the case of cognitive WIFI with an access point. We assume that the transmissions from the access point to all the nodes are perfect but we nevertheless assume that node-to-node transmissions may fail with a probability ρ. This correspond to the case where some nodes may be hidden from each other, *i.e.* they are only connected via the access point. If this probability is independent for each pair of nodes then we are in a random graph model [9]. Such situation may lead to a loss of synchronisation between nodes and the failure of the selection process. To overcome this problem the access point will send a burst to reinforce the burst received from the nodes. We assume that this will ensure the correct synchronisation of the nodes except for the nodes having the supersymbol B_k when the access point has reinforced the super symbol B_{k+1}. For these nodes synchronisation will be achieved again after the access point has spotted sending its burst on the next super symbol. An example of this situation is given in Figure 15.

If one considers that the limitation on $N = 10^6$ is somewhat of artificial, the scheme can easily be extended to $N = 10^{12}$ terminals but with $p = 0.02853$ instead of $p = 0.02$ since the scheme is very sensitive to the tuning of this parameter. The energy cost would be 15.07 burst units per election phase and the collision rate bounded by 0.28. Figure 10 display the average energy cost versus parameter p for $n = 10^6, 10^{12}, 10^{18}$, although the latter values are highly irrealistic in the near future.

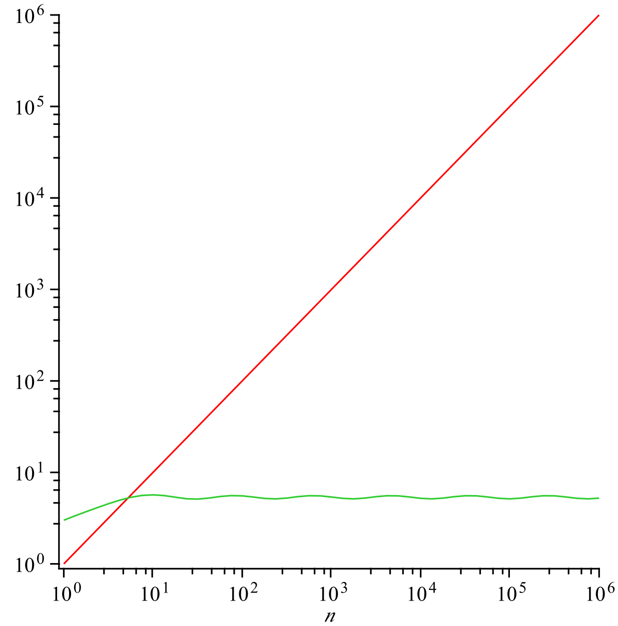

Fig. 9. Theoretical average global energy cost for green leader election (green) versus classic election (red) as a function of n.

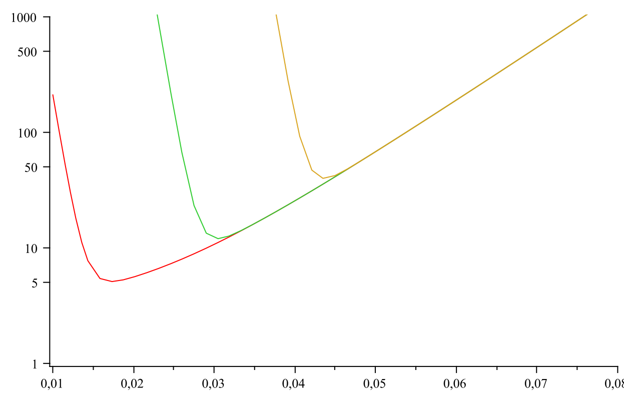

Fig. 10. Theoretical average global energy cost for green leader election versus probability p for $n = 10^6$ (red) classic election (red), $n = 10^{12}$ (green), $n = 10^{18}$ (brown).

V. Numerical and experimental results

We set $p = 0.02$ and $k = 10$. We target $N = 10^6$ and therefore fix $L_N = 3$. We have $\bar{N} = 5.941885894 \times 10^8$. The collision rate is bounded by 0.012. We also have $\bar{N}^{1/10} = 7.540366074$ and $\frac{1}{e}\bar{N}^{1/10} = 2.773945658$ the theoretical maximal average number of transmitters per individual super-symbol.

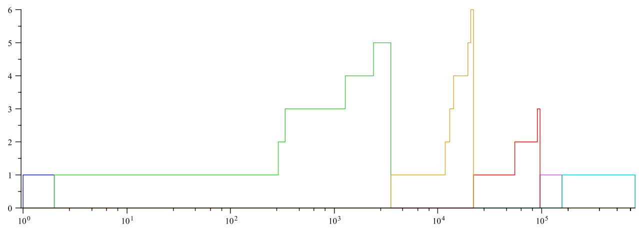

Fig. 11. Simulated number of actual transmitters of first bursts as a function of n (semilog plot).

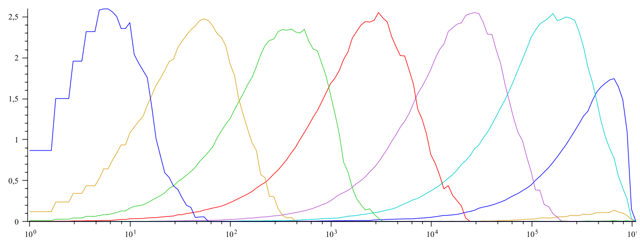

Fig. 12. Simulated average number of transmitters per super-symbol as a function of n (semilog plot). 1,000 independent runs. From left to right, different colors, for super-symbols B_0, B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8 and B_9.

In figures 11 and 12 we display the average number of transmitters on the first super-symbols for all values of n from 1 to $N = 10^6$. The simulation process is the following: we randomly select $X_1, X_2, \ldots, X_N$, and then we assume that the contenders are the n first terminals. Those which actually transmit their first super-symbol are those of rank $i \leq n$ such that $\lfloor k\frac{X_i}{k^{L_N}} \rfloor = \lfloor k\frac{\bar{X}_n}{k^{L_N}} \rfloor$. Notice that the transmitted super-symbol is B_ℓ such that $\ell = \lfloor k\frac{\bar{X}_n}{k^{L_N}} \rfloor$. We let n vary from 1 to $N = 10^6$ for each simulation. In figure 11 we ran only one simulation to illustrate the typical numbers we face in the process. Color changes indicate super-symbol change. We ran 1,000 simulations. Notice that symbol B_9 and B_8 were actually transmitted in the simulation but only once, thus a simulated average of 10^{-3} is unnoticeable on the figure.

Figure 13 displays the simulated average $C(1,n) = \sum_\ell C_\ell(1,n)$ versus the theory. Figure 14 shows the actual standard deviation of $C(1,n)$ obtained from the simulation. We remark that the values displayed in this figure times $\frac{1}{1000}$ naturally hints the error bars of the previous figures.

We now consider nodes communicating with an access point using the LGE mechanism. ρ is the probability that two random nodes can not see each other, we set $N = 10^6$. Figure 16 provides the collision rate as a function of ρ. We observe that even for large ρ, the LGE mechanism provides "good" collision rate. In the same conditions, Figure 17 provides the energy consummed by the stations in the first super-symbol. This energy remains low even large values of ρ if we consider the large number $N = 10^6$ of contenders.

VI. Conclusion

We have described and analyzed a new distributed broadcast leader election algorithm under the innovative performance

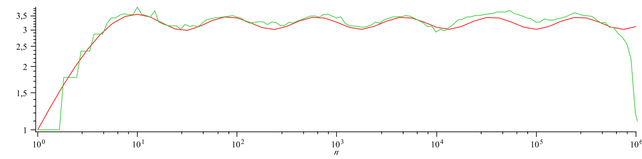

Fig. 13. Simulated mean of $C(1,n)$ (green) and theoretical values versus n from 1 to 10^6, with $k = 10$, $q = 0.98$ and $L_N = 3$.

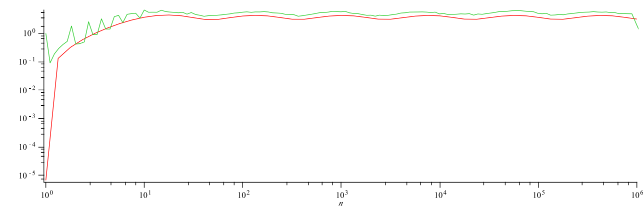

Fig. 14. Simulated standard deviation of $C(1,n)$ (green) and theoretical values versus n from 1 to 10^6, with $k = 10$, $q = 0.98$ and $L_N = 3$.

parameter of the global energy cost. The leader election is limited to the reduction phase; if the reduction phase fails because the number of survivors exceeds two, then a new election phase is executed. However the new algorithm has a global energy cost per *successful* leader election of the order of $(N^{1/k} - 1)\frac{k}{\log N}$ with an average duration of $k \log_k \log N$, where k is an adjustable parameter of the algorithm.

We describe an application to cognitive WiFi, where the green election is used as a systematic medium access protocol. In this case the parameter k is bounded by the ratio between the inter-frame time interval in WiFi and the new mini-slot duration (typically $k \approx 10$).

References

[1] R. G. Gallager, P. A. Humblet, and P. M. Spira "A Distributed Algorithm for Minimum-Weight Spanning Trees". ACM Transactions on Programming Languages and Systems, 1983.

[2] R. M. Metcalfe, D. R. Boggs, Ethernet: distributed packet switching for local computer networks, Communication of the ACM 19 (1976), 395-404.

[3] BS Tsybakov, VA Mikhailov "Random multiple packet access: part-and-try algorithm" Problemy Peredachi Informatsii, 1980.

[4] JC Huang, T Berger "Delay analysis of 0.487 contention resolution algorithms" Communications, IEEE Transactions on IT, 1986

[5] P Flajolet, R Sedgewick "Mellin transforms and asymptotics: finite differences and Rice's integrals" Theoretical Computer Science, 1995

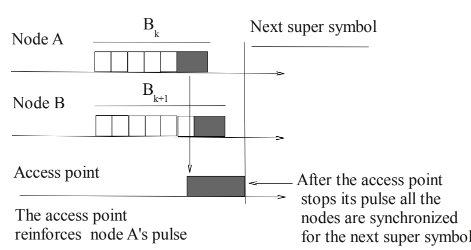

Fig. 15. Synchronisation in cognitive WIFI with an access point.

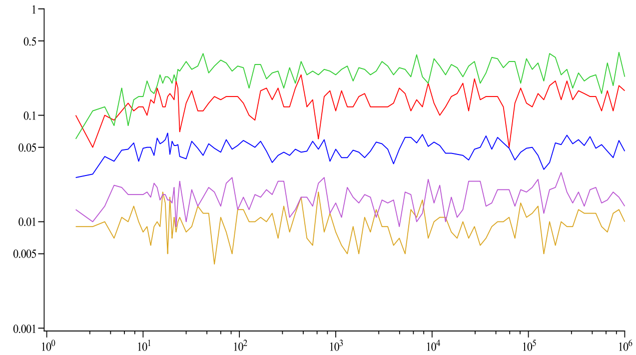

Fig. 16. Mean collision rate in a cognitive WIFI network with an access point as a function of ρ. From the bottom to the top $\rho = 0.0$ (brown), $\rho = 0.1$ (purple) $\rho = 0.4$ (blue) $\rho = 0.8$ (red), $\rho = 1$ (green).

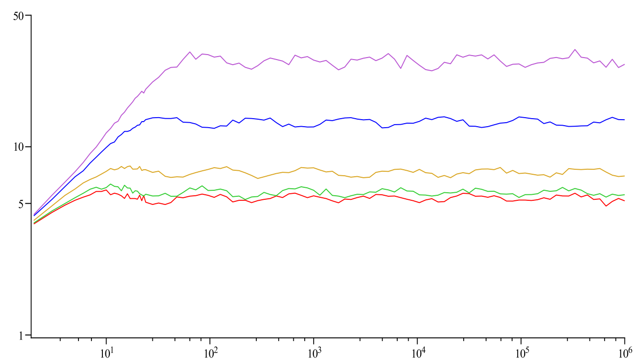

Fig. 17. Mean energy for the first super-symbol in a cognitive WIFI network with a function of ρ. From the bottom to the top $\rho = 0.0$ (red), $\rho = 0.1$ (green) $\rho = 0.4$ (brown) $\rho = 0.8$ (blue), $\rho = 1$ (purple).

[6] J. Fill, H. Mahmoud, W. Szpankowski, "On the Distribution for the Duration of a Randomized Leader Election Algorithm" The Annals of Applied Probability, 1996.

[7] B.P. Crow, I. Widjaja, J.G. Kim, P.T. Sakai, IEEE 802.11, wireless local area networks, IEEE Commun. Mag.,1997.

[8] P Jacquet, W Szpankowski "Analytical depoissonization and its applications", Theoretical Computer Science, 1998

[9] Jacquet, Philippe, and Anis Laouiti. "Analysis of mobile ad-hoc network routing protocols in random graph models." (1999).

[10] PJ Grabner, H Prodinger "Sorting algorithms for broadcast communications: Mathematical analysis" Theoretical computer science, Elsevier, 2002

[11] T Jurdzinski, M Kutylowski, J Zatopianski, "Efficient algorithms for leader election in radio networks", Proceedings of the twenty-first annual symposium on Principles of distributed computing, 2002

[12] P Flajolet, R Sedgewick *Analytic combinatorics*, 2009.

[13] P. Jacquet, P. Mühlethaler, "Cognitive networks: a new access scheme which introduces a Darwinian approach" Wireless Days, Inria research report 7892, 2012.

[14] Heinzelman, W. R., Chandrakasan, A., Balakrishnan, H., Energy-efficient communication protocol for wireless microsensor networks. In System Sciences, 2000. Proceedings of the 33rd Annual Hawaii International Conference on (pp. 10-pp). IEEE.

[15] Gandham, S. R., Dawande, M., Prakash, R., Venkatesan, S. (2003, December). Energy efficient schemes for wireless sensor networks with multiple mobile base stations. In Global telecommunications conference, 2003. GLOBECOM'03. IEEE (Vol. 1, pp. 377-381). IEEE.

[16] Agarwal, M., Gao, L., Cho, J. H., Wu, J. (2005). Energy efficient broadcast in wireless ad hoc networks with hitch-hiking. Mobile Networks and Applications, 10(6), 897-910.

[17] Rajendran, V., Obraczka, K., Garcia-Luna-Aceves, J. J. Energy-efficient, collision-free medium access control for wireless sensor networks. Wireless Networks, 12(1), 63-78, 2006.

[18] Meghan, G. U. N. N., Simon, G. M. (2009). A comparative study of medium access control protocols for wireless sensor networks. Int'l J. of Communications, Network and System Sciences, 2(8), 695-703.

[19] Niewiadomska-Szynkiewicz, E., Kwasniewski, P., Windyga, I. (2009). Comparative study of wireless sensor networks energy-efficient topologies and power save protocols. Journal of Telecommunications and Information Technology, 3, 68-75.